TWO FACTOR ANALYSIS OF VARIANCE

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THERE ARE TWO IMPORTANT considerations when calculating the two factor ANOVA: first, it is necessary to lay out the data correctly and second, the correct error terms must be chosen for the variance ratios. In this study notes three different types of two factor ANOVA are dealt with: the two factor independent measures ANOVA where both the factors, *A* and *B*, are independent measures; the two factor mixed design ANOVA where Factor *A* is independent measures and Factor *B* is repeated measures, and the two factor repeated measures ANOVA where both Factor *A* and Factor *B* are repeated measures.

The two factor independent measures ANOVA

The simplest two factor ANOVA to calculate is where both factors are independent measures. Here the between conditions variance has to be separated into that arising from Factor *A*, Factor *B* and the interaction $A \times B$, as in all two factor ANOVAs. As there are individual differences in all sums of squares calculations we can use the within conditions variance as the error term for all three variance ratios. This makes the calculations relatively easy. We, therefore, complete the following ANOVA summary table.

THE ANOVA SUMMARY TABLE

Source of variation	Degrees of freedom	Sums of squares	Mean square	Variance ratio (*F*)	Probability
Factor *A*	df_A	SS_A	MS_A	F_A	p_A
Factor *B*	df_B	SS_B	MS_B	F_B	p_B
Interaction $A \times B$	$df_{A \times B}$	$SS_{A \times B}$	$MS_{A \times B}$	$F_{A \times B}$	$p_{A \times B}$
Error (Within conditions)	df_{error}	SS_{error}			
Total	df_{total}	SS_{total}			

The results table

Organising the results table is important for all ANOVAs but which factor we choose as the rows and which as the columns is not as crucial for the two factor independent measures ANOVA as for the other types of two factor ANOVA, but it is important to get the various totals of the different conditions and combination of conditions correct. The following data layout is a good example to use for clarity and organisation.[11]

THE RESULTS TABLE

Factor A	Factor B			
	Condition B_1	Condition B_2	...	Condition B_b
Condition A_1	X_1	$X_{...}$	...	$X_{...}$
	X_2	$X_{...}$	...	$X_{...}$
	$\vdots$	$\vdots$	...	$\vdots$
	X_n	$X_{...}$	...	$X_{...}$
	$T_{A_1B_1}$	$T_{A_1B_2}$	...	$T_{A_1B_b}$	T_{A_1}
Condition A_2	$X_{...}$	$X_{...}$	...	$X_{...}$
	$X_{...}$	$X_{...}$	...	$X_{...}$
	$\vdots$	$\vdots$		$\vdots$
	$X_{...}$	$X_{...}$	...	$X_{...}$
	$T_{A_2B_1}$	$T_{A_2B_2}$	...	$T_{A_2B_b}$	T_{A_2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Condition A_a	$X_{...}$	$X_{...}$		$X_{...}$
	$X_{...}$	$X_{...}$		$X_{...}$
	$\vdots$	$\vdots$		$\vdots$
	$X_{...}$	$X_{...}$		X_{abn}
	$T_{A_aB_1}$	$T_{A_aB_2}$	...	$T_{A_aB_b}$	T_{A_a}
	T_{B_1}	T_{B_2}		T_{B_b}	$\sum X$

The formulae for calculation

Degrees of freedom:

$$df_A = a - 1$$ where a is the number of condition of Factor A.

$$df_B = b - 1$$ where b is the number of conditions of Factor B.

$$df_{A \times B} = (a - 1)(b - 1)$$

$$df_{error} = ab(n - 1)$$ where n is the number of scores in an AB condition.

$$df_{total} = N - 1$$ where N is the total number of scores in the data.

Sums of squares:

$$SS_{total} = \sum X^2 - \frac{(\sum X)^2}{N}$$

$$SS_A = \frac{\sum T_A^2}{nb} - \frac{(\sum X)^2}{N}$$ where $\sum T_A^2$ is $T_{A_1}^2 + T_{A_2}^2 + \ldots + T_{A_a}^2$

$$SS_B = \frac{\sum T_B^2}{na} - \frac{(\sum X)^2}{N}$$ where $\sum T_B^2$ is $T_{B_1}^2 + T_{B_2}^2 + \ldots + T_{B_b}^2$

$$SS_{A \times B} = \frac{\sum T_{AB}^2}{n} - \frac{(\sum X)^2}{N} - SS_A - SS_B$$ where $\sum T_{AB}^2$ is $T_{A_1 B_1}^2 + T_{A_1 B_2}^2 + \ldots + T_{A_a B_b}^2$

$$SS_{error} = SS_{total} - SS_A - SS_B - SS_{A \times B}$$

(There is an alternative formula for SS_{error}:

$$SS_{error} = SS_{with.conds} = \sum X^2 - \frac{\sum T_{AB}^2}{n}$$

Both formulae should give the same answer.)

Mean square:

$$MS_A = \frac{SS_A}{df_A}$$

$$MS_B = \frac{SS_B}{df_B}$$

$$MS_{A\times B} = \frac{SS_{A\times B}}{df_{A\times B}}$$

$$MS_{error} = \frac{SS_{error}}{df_{error}}$$

Variance ratio:

$$F_A(df_A, df_{error}) = \frac{MS_A}{MS_{error}}$$

$$F_B(df_B, df_{error}) = \frac{MS_B}{MS_{error}}$$

$$F_{A\times B}(df_{A\times B}, df_{error}) = \frac{MS_{A\times B}}{MS_{error}}$$

The *F* values are then compared to the table values (Table A.1 in the Appendix) at the chosen level of significance.

(The above calculations are based on equal numbers of scores, *n*, in each of the *AB* conditions. It is possible to perform this analysis with unequal numbers of scores in each condition, as with the single factor independent measures ANOVA.

A worked example

An expanding company wanted to know how to introduce a new type of machine into the factory. Should it transfer staff working on the old machine

to operate it or employ new staff who had not worked on any machine before? A researcher selected 12 staff who had experience of the old machine and 12 staff who had no such experience. Half the participants from each group were allocated to the new machine and half to the old machine. The number of errors made by the participants over a set time period was measured. These errors are shown below.

Experience on old machine	Machine	
	Old	New
Novice	4	5
	5	6
	7	5
	6	6
	8	5
	5	6
Experienced	1	8
	2	9
	2	8
	3	8
	2	7
	3	9

What are the effects of the two factors *experience on old machine* and *type of machine* on the dependent variable *number of errors*?

Both factors are independent measures as a participant took part in only one experience/machine condition. I will label *experience on old machine* as Factor A, with two conditions ($a = 2$) 'novice' (A_1) and 'experienced' (A_2), and *type of machine* as Factor B, also with two conditions ($b = 2$), 'old machine' (B_1) and 'new machine' (B_2). There are four AB conditions each with six participants ($n = 6$), giving twenty-four participants in all ($N = 24$).

Factor A	Factor B		
	B_1	B_2	
A_1	4	5	
	5	6	
	7	5	
	6	6	
	8	5	
	5	6	
	$T_{A_1B_1} = 35$	$T_{A_1B_2} = 33$	$T_{A_1} = 68$
A_2	1	8	
	2	9	
	2	8	
	3	8	
	2	7	
	3	9	
	$T_{A_2B_1} = 13$	$T_{A_2B_2} = 49$	$T_{A_2} = 62$
	$T_{B_1} = 48$	$T_{B_2} = 82$	$\sum X = 130$

Degrees of freedom:

$$df_A = a - 1 = 2 - 1 = 1$$
$$df_B = b - 1 = 2 - 1 = 1$$
$$df_{A \times B} = (a - 1)(b - 1) = (2 - 1)(2 - 1) = 1$$
$$df_{error} = ab(n - 1) = 2 \times 2 \times (6 - 1) = 20$$
$$df_{total} = N - 1 = 24 - 1 = 23$$

Sums of squares:

$$SS_{total} = \sum X^2 - \frac{(\sum X)^2}{N} = (4^2 + 5^2 + \ldots + 7^2 + 9^2) - \frac{130^2}{24}$$
$$= 127.83$$

$$SS_A = \frac{\sum T_A^2}{nb} - \frac{(\sum X)^2}{N} = \frac{68^2 + 62^2}{6 \times 2} - \frac{130^2}{24} = 1.50$$

$$SS_B = \frac{\sum T_B^2}{na} - \frac{(\sum X)^2}{N} = \frac{48^2 + 82^2}{6 \times 2} - \frac{130^2}{24} = 48.17$$

$$SS_{A \times B} = \frac{\sum T_{AB}^2}{n} - \frac{(\sum X)^2}{N} - SS_A - SS_B$$

$$= \frac{35^2 + 33^2 + 13^2 + 49^2}{6} - \frac{130^2}{24} - 1.50 - 48.17$$

$$= 60.16$$

$$SS_{error} = SS_{total} - SS_A - SS_B - SS_{A \times B}$$

$$= 127.83 - 1.50 - 48.17 - 60.16 = 18.00$$

Mean square:

$$MS_A = \frac{SS_A}{df_A} = \frac{1.50}{1} = 1.50$$

$$MS_B = \frac{SS_B}{df_B} = \frac{48.17}{1} = 48.17$$

$$MS_{A \times B} = \frac{SS_{A \times B}}{df_{A \times B}} = \frac{60.16}{1} = 60.16$$

$$MS_{error} = \frac{SS_{error}}{df_{error}} = \frac{18.00}{20} = 0.90$$

Variance ratio:

$$F_A(1,20) = \frac{MS_A}{MS_{error}} = \frac{1.50}{0.90} = 1.67$$

$$F_B(1,20) = \frac{MS_B}{MS_{error}} = \frac{48.17}{0.90} = 53.52$$

$$F_{A \times B}(1,20) = \frac{MS_{A \times B}}{MS_{error}} = \frac{60.16}{0.90} = 66.84$$

THE ANOVA SUMMARY TABLE

| Source of variation	Degrees of freedom	Sums of squares	Mean square	Variance ratio (F)	Probability
Factor A	1	1.50	1.50	1.67	$p > 0.05$
Factor B	1	48.17	48.17	53.52	$p < 0.01$
$A \times B$	1	60.16	60.16	66.84	$p < 0.01$
Error	20	18.00	0.90		
Total	23	127.83			

From the tables of the F distribution (A.3 in the Appendix), $F(1,20)$ = 4.35 at p = 0.05 and $F(1,20)$ = 8.10 at p = 0.01. We can conclude that the effect of *experience on an old machine* is not significant at p = 0.05 ($F(1,20)$ = 1.67), the effect of *type of machine* ($F(1,20)$ = 53.52) and the interaction ($F(1,20)$ = 66.84) are both highly significant ($p < 0.01$).

We can examine the interaction by calculating the mean values. The table of means is shown below:

Experience on old machine	Machine	
	Old machine	New machine
Novice	5.83	5.50
Experienced	2.17	8.17

These values are plotted in Figure 1.1. The first point to note is that the lines are not parallel so we have further evidence of the interaction. Notice that the experienced workers, not surprisingly, made fewest errors on the old machine. However, they made most errors on the new machine. This looks like a case of negative transfer, where previously learnt skills can be a hindrance rather than a help. An example of this occurs when a visitor to Britain, experienced in a left-hand drive car, reaches down to change gear with the wrong hand when driving a right-hand drive car. The novice workers appear to perform with equal accuracy on both machines.
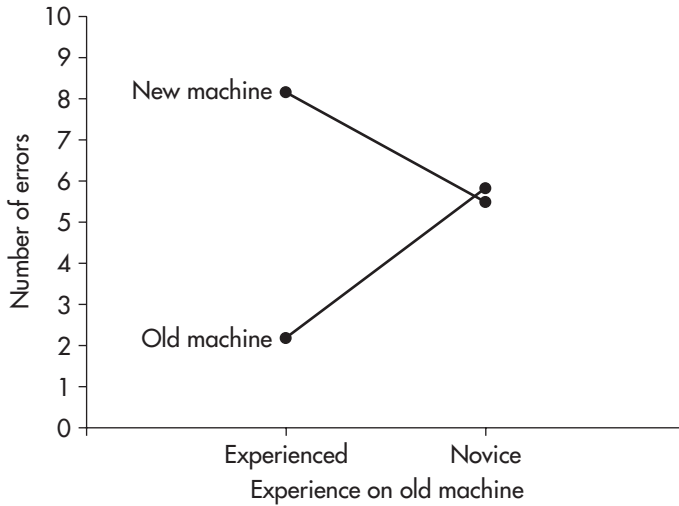
FIGURE 1.1 The interaction of experience and machine on the number of errors

In this case the interaction is quite clear. However, for illustration the simple main effects will be calculated for the effect of *type of machine* on the two levels of experience. In the two factor independent design ANOVA the error term is once again the single error term from the summary table: $MS_{error} = 0.90$, $df_{error} = 20$. This error term is used in all the simple main effects.

The simple main effect of *type of machine* on the novice operators, B at A_1:

$$SS_{B\ at\ A_1} = \frac{\sum T_{A_1B}^2}{n} - \frac{T_{A_1}^2}{bn} = \frac{35^2 + 33^2}{6} - \frac{68^2}{2 \times 6} = 0.33$$

$df_{B\ at\ A_1} = b - 1 = 2 - 1 = 1$ (as it is the effect of B and B has
2 conditions)

$$MS_{B\ at\ A_1} = \frac{SS_{B\ at\ A_1}}{df_{B\ at\ A_1}} = \frac{0.33}{1} = 0.33$$

$$F_{B\ at\ A_1} = \frac{MS_{B\ at\ A_1}}{MS_{error}} = \frac{0.33}{0.90} = 0.37$$

with degrees of freedom $df_{B\ at\ A_1} = 1$ and $df_{error} = 20$.

From the F distribution tables we know that $F(1,20) = 4.35$ at $p = 0.05$, so we can conclude, as the calculated value of F is smaller, that we have not found an effect of *type of machine* on the novice operators.

The simple main effect of *type of machine* on the experienced operators, B at A_2:

$$SS_{B \, at \, A_2} = \frac{\sum T^2_{A_2 B}}{n} - \frac{T^2_{A_2}}{bn} = \frac{13^2 + 49^2}{6} - \frac{62^2}{2 \times 6} = 108.00$$

$$df_{B \, at \, A_2} = b - 1 = 2 - 1 = 1 \text{ (as it is the effect of } B \text{, and } B \text{ has}$$
$$2 \text{ conditions)}$$

$$MS_{B \, at \, A_2} = \frac{SS_{B \, at \, A_2}}{df_{B \, at \, A_2}} = \frac{108.00}{1} = 108.00$$

$$F_{B \, at \, A_2} = \frac{MS_{B \, at \, A_2}}{MS_{error}} = \frac{108.00}{0.90} = 120.00$$

with degrees of freedom $df_{B \, at \, A_1} = 1$ and $df_{error} = 20$.

From the F distribution tables we know that $F(1,20) = 8.10$ at $p = 0.01$, so we can conclude, as the calculated value of F is considerably larger, that we have a found a highly significant effect of *type of machine* on the experienced operators.

The simple main effects usually explain the cause of an interaction but we can perform post hoc tests such as the Tukey or Scheffé tests if we wish. We need to be careful to select the appropriate comparison and the correct error term although it is particularly easy with the independent measures design as we use just the one error term.

A.1 Critical values of the *F* distribution

0.05 Level of significance

df 2	1	2	3	4	5	6	7	8	9	10	20	∞
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	248.01	254.32
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.45	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.66	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.80	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.56	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	3.87	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.44	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.15	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.77	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.65	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.54	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.46	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.39	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.33	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.28	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.23	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.19	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.16	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.12	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.10	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.07	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.05	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.03	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.01	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	1.99	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	1.97	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	1.96	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	1.94	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	1.93	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	1.84	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.75	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.66	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.57	1.00

*df*1 is the column header spanning columns 1 through ∞.

The calculated value of *F* must be larger than or equal to the table value for significance.

A.1 Critical values of the *F* distribution (continued)

0.01 Level of significance

df 2	*df* 1											
	1	2	3	4	5	6	7	8	9	10	20	∞
1	4052.2	4999.5	5403.3	5624.6	5763.7	5859.0	5928.3	5981.6	6022.5	6055.8	6208.7	6366.0
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.45	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.34	27.23	26.69	26.12
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.02	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.55	9.02
6	13.74	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.40	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.16	5.67
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.36	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	4.81	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.41	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.10	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	3.86	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.66	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.51	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.37	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.26	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.16	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.08	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.00	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	2.94	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	2.88	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	2.83	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	2.78	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	2.74	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	2.70	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.66	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.63	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.60	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.57	2.03
30	7.58	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.55	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.37	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.20	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.03	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	1.88	1.00

The calculated value of *F* must be larger than or equal to the table value for significance.